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Relativistic electron precipitation events driven by electromagnetic ion-cyclotron waves

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We adopt a canonical approach to describe the stochastic motion of relativistic belt electrons and their scattering into the loss cone by nonlinear EMIC waves. The estimated rate of scattering is sufficient to account for the rate and intensity of bursty electron precipitation. This interaction is shown to result in particle scattering into the loss cone, forming ∼10 s microbursts of precipitating electrons. These dynamics can account for the statistical correlations between processes of energization, pitch angle scattering, and relativistic electron precipitation events, that are manifested on large temporal scales of the order of the diffusion time ∼tens of minutes.
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I. INTRODUCTION

Proton-mode whistlers, or electromagnetic ion-cyclotron (EMIC) waves, are frequently observed near the vicinity of the equatorial plasmapause, where they are thought to be driven by a ring current anisotropy.[1,2] The waves exhibit a periodic structure of a sequence of discrete wave packets with a repetition period of 10 s at frequency ∼1.5–2 Hz. Direct observations show that EMIC waves propagate along the direction of the ambient magnetic field, and broadband wave amplitudes can reach ∼1–10 nT.[3–7]

Pitch angle scattering caused by resonant interaction with EMIC waves is thought to be the most likely mechanism for relativistic electron precipitation (REP) events, and observations of REP events consistent with this mechanism have been made,[8–10] and modelled as quasi-linear diffusion.[11] A review of the observations is given by Millan and Thorne.[12] A number of investigators have addressed the theory and simulation of nonlinear EMIC wave-particle interactions, e.g., Albert and Bortnik,[13] An *et al.*[14] Of these, the approach of An *et al.* is quite promising. Using a two-wave model, the authors study the impact of wave amplitude modulation on the interaction of electrons and EMIC waves. Though, quoting An *et al.*,[14] "this is the first step toward understanding the interaction between electrons and a realistic EMIC wave packet," the approach is reasonable because it allows to determine the conditions for which phases are randomized and as a consequence, a kinetic description of the motion can be applied.[15] We will study stochastic motion of relativistic electrons driven by nonlinear EMIC waves. We show the dynamics are ergodic with phase mixing, so that the evolution of spectra and observable quantities obeys a kinetic equation. This gives a way to calculate the rates of energization and particle scattering, and evaluate REP events.

We consider the dynamics of relativistic electrons resonantly interacting with nonlinear EMIC waves. In Sec. II, we derive the equations of motion and find the conditions under which the motion becomes chaotic. The statistical aspects of the dynamics are treated by analytical and numerical means in Sec. III. The effect of external noise in modifying the motion is also examined. In Sec. IV, we deal with the problem of pitch angle scattering and REP events. The principal conclusions are given in Sec. V.

II. PARTICLE MOTION

We will consider the resonant interaction of relativistic electrons with field-aligned EMIC waves in terms of the canonical approach. According to Khazanov *et al.*,[16] the problem can be put in the form of a Hamiltonian

$$H(z,p_z;\theta,I;t) = H_0(p_z,I) + \sqrt{2m\omega_B I}H_0^{-1}\cdot B^w(z,\theta,t)/k_z, \tag{1}$$

$$H_0(p_z,I) = \sqrt{m^2+p_z^2+2m\omega_B I}, \quad \omega_B = B/m, \tag{2}$$

where ω_B is the gyrofrequency. The Hamiltonian is given by two pairs of canonical variables, (p_z,z), (θ,I) on a smooth manifold M. M is the usual 2-D phase space: a point in M is given by 4-tuple of real numbers. We have employed here, and throughout this paper, the system of units in which the speed of light $c=1$ and the electron charge $|e|=1$, and we have chosen a Cartesian spatial coordinate system whose z axis is directed along the external magnetic field B. The action-angle (I,θ) pair is immediately related with the gyromotion and magnetic moment, and any fixed value of H_0, $H_0=E$, determines the particle energy E, $B^w(z,\theta,t)$ is the wave magnetic field

$$B^w = B^w(\varepsilon t,\varepsilon z)\cos\psi, \quad \psi = k_z z + s\theta - \omega t, \tag{3}$$

with slowly varying amplitude $B^w(\varepsilon t,\varepsilon z)$, and ψ is the phase of the particle in the wave. The smooth periodic function $B^w(\varepsilon t,\varepsilon z)$ describes the repetitive structure of the wave envelope and satisfies the conditions

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$$B^w(t+T, z+L) = B^w(t,z), \tag{4}$$

$$\varepsilon = \frac{\partial B^w}{\partial t}\cdot\frac{1}{\omega B^w} \sim (\omega T)^{-1}, \quad \frac{\partial B^w}{\partial z}\cdot\frac{1}{kB^w} \sim (kL)^{-1}. \tag{5}$$

Here, ε is a small parameter, the ratio of the oscillation period $2\pi/\omega$ (or $2\pi/k$) to the time (space) scale $(T,\ L)$ over which the envelope varies. So, we assume that the ambient magnetic field varies slowly over one wavelength $\sim\frac{1}{Bk_z}\partial B/\partial z\,(\sim\varepsilon^2)$, and that the magnitude of the wave field is sufficiently small, $B^w/B(\sim\varepsilon)$. Resonant wave-particle interaction occurs whenever the resonance condition

$$\dot{\psi} = k_z\dot{z} + s\dot{\theta} - \omega = 0, \quad s = -1, \tag{6}$$

is met. Near resonance, the variables ψ and p_z vary slowly compared to time period of oscillations, and the following conditions of the adiabatic approach are valid

$$\dot{\psi} \ll \omega\psi, \quad \dot{I} \ll \omega I, \quad \dot{p}_z \ll \omega p_z \tag{7}$$

These conditions allow us to write equations of motion associated with (1) as

$$\dot{p}_z = k_z\sqrt{2m\omega_B I}H_0^{-1}\cdot(B^w(\varepsilon t, z)/k_z)\sin\psi, \tag{8}$$

$$\dot{I} = s\sqrt{2m\omega_B I}H_0^{-1}\cdot(B^w(\varepsilon t, z)/k_z)\sin\psi, \tag{9}$$

$$\dot{z} = p_z/H_0, \quad \dot{\theta} = \omega_B m/H_0. \tag{10}$$

These equations retain only the leading terms, all of the perturbation terms average to zero, and the small parameter ε automatically keeps track of the ordering. It stands to reason that the phase flow (8)–(10) conserves the invariant of motion

$$-sp_z + k_z I = \text{const.}, \tag{11}$$

where const. is a constant independent of time given by the initial resonance condition. Proton-mode whistlers propagate at frequencies below the proton gyrofrequency and can interact with relativistic electrons through the anomalous electron gyroresonance, $k(p/E) = \omega_B(m/E)$. This allows us to evaluate the value of p_z at the resonance with the principal mode as

$$p_r/m = (\omega_B/\omega)v_p, \tag{12}$$

where v_p is the phase velocity of wave, which is close to the Alfven speed. For the majority of EMIC wave events, the resonant energy E_r for electrons was found to be about 2 MeV,[12] so we will take $p_r = E_r$ with a sufficient accuracy, $E_r - p_r/E_r \sim m^2/2p_r^2$. In view of (12), we write the invariant of motion (11) as

$$p_z + \frac{2m\omega_B I}{2p_r} = p_r, \quad p_z \in (m, p_r). \tag{13}$$

This invariant leads to the restriction of dynamics onto a reduced phase space, acceptable coordinates of which are the canonically conjugated pair (ψ, u), where ψ is the phase variable and u is a new action variable. Even with these simplifications, we can reduce this motion to quadratures only if the perturbation has a trivial form of monochromatic wave propagating along the direction of the ambient magnetic field. In the general case, this nonautonomous nonlinear dynamical system is non-integrable, and the measure of its regular motion is equal to zero.

Consider the extended phase space on which the periodic Hamiltonian is given. In this case, it is sufficient to describe the motion in some time interval (t_0, t_0+T), such as $g^1(\psi_0, u_0) = (\psi_1, u_1)$, where (ψ_0, u_0) is the initial state of the system and g^1 is the map at one period.[17] In this way, we can define a map $g^n = (g^1)^n$ of the phase plane onto itself. The map preserves the phase volume in virtue of Liouville's theorem.

For the Hamiltonian (1) and its invariant of motion, a standard representation can be written in the form of a set of nonlinear difference equations[16]

$$\begin{aligned} u_{n+1} &= u_n + Q\sin\psi_n, \\ \psi_{n+1} &= \psi_n + F(u_{n+1}) \mod 2\pi, \end{aligned} \tag{14}$$

where u_n and ψ_n are taken at $t = nT$, Q is the control parameter that defines the intensity of wave-particle interaction, and the function $F(u_{n+1})$ describes the shift of phase acquired by a particle, with $n \in Z$, where Z is the set of all integers.

In order to find the explicit form of these equations, we need a way to define the wave form of the envelope of the nonlinear EMIC wave. There exist two equivalent representations, the first of which describes these waveforms by solutions of nonlinear wave equations, usually the nonlinear Schrödinger equation, the applicability of which to Alfven, EMIC, and whistler waves is well-grounded.[18,19] Another typical representation is to write the wave field in the form of a nonlinear wave packet[20]

$$B^w(t,z) = (B_0^w/\sqrt{\Delta kL})\sum_{n\in Z}\delta(z/L - n)\cos\psi, \quad L = v_A T, \tag{15}$$

where B_0^w is the peak value of wave amplitude, Δk is the width of wave packet, and the relation $\sum_{n\in Z} e^{inx} = 2\pi\sum_{n\in Z}\delta(x - 2\pi n)$, where $\delta(\cdot)$ is the Dirac delta function, has been used in writing (15). The equivalence of these representations in describing wave-particle resonance interaction has been shown by the authors Khazanov *et al.*[16]

To qualitatively understand the behavior the system, we first investigate the dynamics of particles in the single mode of nonlinear wave. Assuming that the wave perturbation is sufficiently small, so that the deviation of the p_z-momentum of particle from its resonance value makes a small change of the phase of particle in the wave field, we write Eqs. (6) and (8) as follows

$$\dot{p}_z = -\omega_f(\omega_f/k_z)p_r\sin\psi,\ \dot{\psi} = k_z(p_z - p_r)/E_r, \tag{16}$$

putting the values of quantities at the resonance, $k_z(p/E)_r = \omega_B m/E_r$. Equations (16) describe the modulated oscillation of particle whose bounce-frequency is given by

$$\omega_f = k_z \sqrt{ub/\sqrt{\Delta k L}}, \quad u = \sqrt{2m\omega_B I}/p_r, \quad b = B_0^w/B. \quad (17)$$

Following the approach of Chirikov,[21] we use the general notions of overlapping resonances to predict the stochastic nature of the motion. In this approach, the overlap of resonances is closely associated with the parametric resonance, $\omega_f \simeq \delta\omega$, between the bounce-frequency and the difference frequency in wave packet. The importance of the bounce resonance has been noted by Khazanov *et al.*[16] As indicated by (6) and (12), the frequency spacing in the wave $\delta\omega = \delta k (p/E)_r$, $\delta k \simeq 1/L$. It stands to reason, the transition from regular motion to stochastic one is possible only if the condition

$$u\sqrt{(\omega/\Delta\omega)\omega T b^2}\omega T \geq 1 \quad (18)$$

is satisfied. This result is the consequence of the bounce resonance.

Our interest now is to inspect the motion of relativistic electrons driven by nonlinear EMIC waves. Substituting (15) into (10) and integrating the resulting equation over the space scale L yields a map in the standard form of a closed pair of nonlinear difference equations

$$g^n : \begin{aligned} u_{n+1} &= u_n + Q \sin\psi_n, \\ \psi_{n+1} &= \psi_n + (\omega T u_{n+1}^2/2)/(1 - u_{n+1}^2/2) \bmod 2\pi, \end{aligned} \quad (19)$$

where the u-variable and the control parameter Q are given by

$$u \equiv \sqrt{2m\omega_B I}/p_r, \quad Q = \sqrt{\frac{\omega}{\Delta\omega}\omega T b^2} = \sqrt{\omega^2 P T}, \quad b = B_0^w/B, \quad (20)$$

and $P = b^2/\Delta\omega$ is the normalized wave power, $\Delta\omega$ is the width of frequency spectrum. These equations, the map g^n, describe the change in the transverse momentum of particle in multiple encounters with the nonlinear wave packet. In deriving g^n, we have used the invariant of motion, and the first of Hamilton's equations (10), $dt = dz(E/p_z)$. The equations (19) are valid in a range of u, $u \leq u_b$, bounded by the invariant of motion (13),

$$u_b = \sqrt{2(1 - m/p_r)}, \quad u_b = \sqrt{3/2}, \quad (21)$$

where u_b is the upper bound of u-values. We have integrated these equations numerically with an initial condition, using typical parameters for wave-particle interaction, namely, $\omega_B/2\pi = 8 \times 10^3$ Hz, $B = 300$ nT, $b = (1-3) \times 10^{-2}$, $T = 40$ s, and $\Delta\omega/\omega = 0.25$, $v_A = 3 \times 10^5$ m/s, and $Q \sim 0.1$. Fig. 1 illustrates the type of dynamics.

III. STATISTICAL PROPERTIES OF THE DYNAMICS

Let us consider a pair (M, g^n), where M is a smooth manifold. It is known[17] that the geometrical structure of M and dynamics of g^n on M are intimately closed. Indeed, while all smooth curves have topological dimension one, a dimension of phase curve constructed by g^n approximates two. In

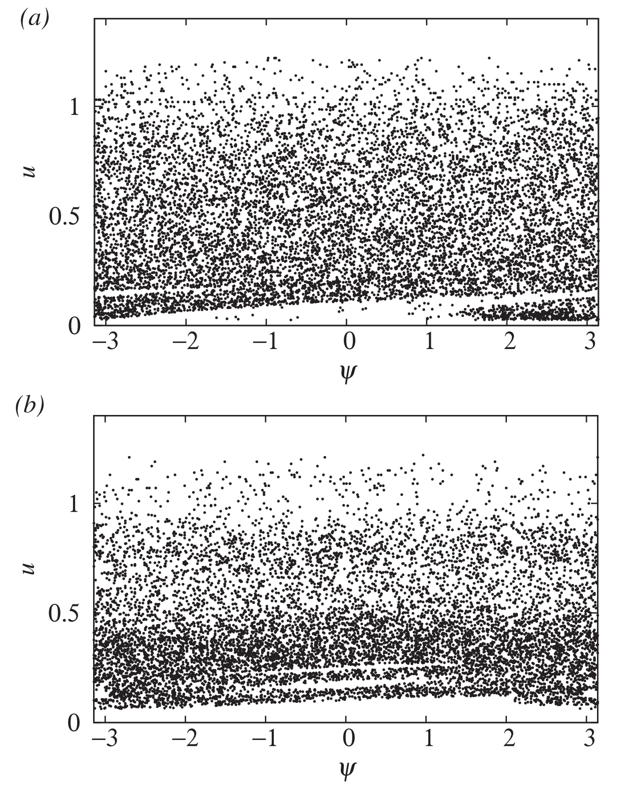

FIG. 1. Single phase trajectory of length 2×10^4 for the map g^n with (a) $Q = 0.1$ and (b) $Q = 0.04$.

such cases, we speak of the stochastic dynamics on the strange attractor (SA). In order to see it, one examines the local topology of M. Denote by

$$J = \frac{\partial(u_{n+1}, \psi_{n+1})}{\partial(u_n, \psi_n)} \quad (22)$$

the Jacobi matrix of the map, the eigenvalues of which are given by the relations

$$\begin{aligned} \det J &= \lambda_1 \cdot \lambda_2 = 1, \\ \operatorname{tr} J &= \lambda_1 + \lambda_2 = 2 + \omega T Q u/(1 - u^2/2)^2, \end{aligned} \quad (23)$$

where $\det J$ and $\operatorname{tr} J$ denote the determinant and the trace of the matrix, respectively. The Jacobian of (22) is equal to one, and therefore (ψ, u) is the canonical pair of variables of area-preserving map g^n. According to Arnold,[17] the condition

$$|\operatorname{tr} J| - 1 \geq 2 \quad (24)$$

implies that phase space has topological structure of hyperbolic type, so that any phase trajectory near a hyperbolic point diverges from it. This property is closely related to the local instability of phase trajectories.[20] To show it, we study the behavior of the system near the lower boundary, u_a, of u-values in the vicinity of fixed points given by the equation, $\sin\psi_0 = 0$. In this case, the map g^n reduces to the well-known circle map[22]

$$\psi_{n+1} = \psi_n + \omega T Q u_a \sin\psi_n \bmod 2\pi, \quad (25)$$

where we put $u_a^2/2 \ll 1$ in accordance with numerical results. The dynamics of the map are stochastic provided the multiplicator of the map is positive

$$\left|\frac{\delta\psi_{n+1}}{\delta\psi_n} - 1\right| = \omega T Q u_a (\geq 1). \tag{26}$$

This process, as we will see later on, leads to the decay of phase correlations and to phase mixing. Then, we apply the topological condition (24) to the relation (23) to obtain an expression

$$u_a = 1/\omega T Q, \tag{27}$$

that agrees with the two qualitative estimates (26) and (18). Expression (27) determines the lower bound of the set of u-values, which is well confirmed numerically. Figure 1 shows that all points of phase trajectory belong to a compact set, $u \in (u_a, u_b)$ and $\psi \in (-\pi, \pi)$; that is, any phase trajectory is bounded, though it diverges locally. The combination of these properties, global stability and local instability, brings about chaotic dynamics. For a finite system phase space, phase trajectories cannot diverge more than a characteristic size of the phase space due to local instabilities. Denote by

$$d_n = \sqrt{(\psi_n - \psi_n')^2 + (u_n - u_n')^2}, \tag{28}$$

the distance between two points in phase space belonging to nearby trajectories at a moment of time n. The evolution of this quantity in time is shown in Fig. 2. We can also describe this process by means of Liapunov exponents. Applying topological condition (24) to (23) yields $\ln\lambda_1 = (3+\sqrt{5})/2$. The positive Liapunov exponent characterizes the mean rate of local instability, and, as a consequence, the rate of the loss of information about initial conditions, the Kolmogorov entropy h_K, $h_K = \ln\lambda_1$ and the correlation time, $\tau_c \sim 1/h_K$. In this way, the relation establishes the interdependence between the statistical characteristic (h_K), and the properly dynamical one ($\ln\lambda_1$). Note that any integrable Hamiltonian system has all Lyapunov exponents equal to zero, and its trajectory is completely determined by initial conditions. To examine its statistical properties, we define a related quantity, the correlation function $C(i)$,

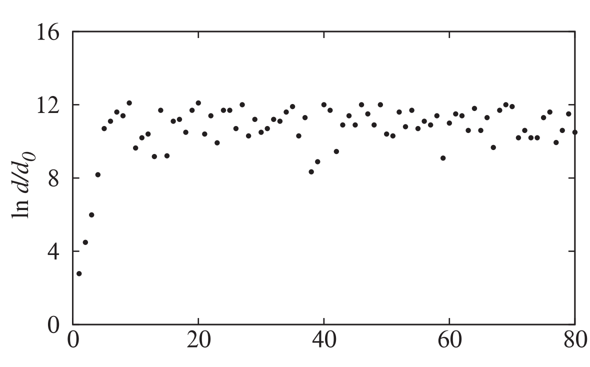

FIG. 2. Development of local instability.

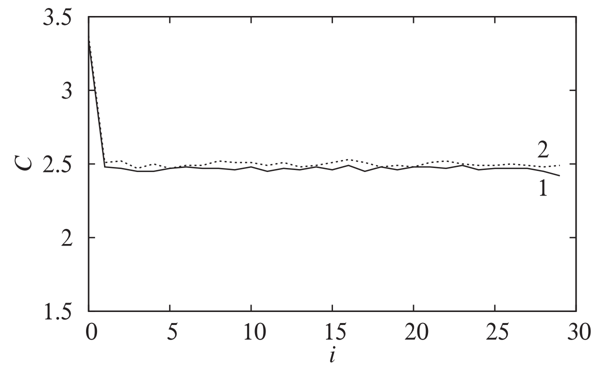

FIG. 3. Correlation function $C(i)$ on i (1) without noise and (2) with noise. $Q = 0.1$.

$$C(i) = (1/N)\sum_{n\in N}\psi(n)\psi(n+i),$$

where i is the step lag and N is the total number of steps of iterations. Numeral investigation of the $C(i)$ gives the results shown in Fig. 3. Figure 3 shows a complete decorrelation of the motion in one mapping period. Note that the Fokker–Planck–Kolmogorov (FPK) description is valid only when the function $C(i)$ falls off rapidly with the number of mapping iterations. Another important characteristic is fractal dimension, d_f, which is related to the spectrum of Liapunov exponents as

$$d_f = 1 - \ln\lambda_1/\ln\lambda_2, \quad d_f = 2. \tag{29}$$

This d_f is equal to the dimension of phase plane; that is, the phase trajectory evenly fills all accessible phase space. This means that the probability density of states tends to the invariant distribution

$$\rho(\psi, u) = 1/2\pi(u_b - u_a). \tag{30}$$

The motion on any SA is random over a wide range of Q due to the global stability of the SA, on which all means (observables) are stable independent of any (reasonable) initial conditions. All this taken together implies that the present system belongs to the class of K-systems.[20] The (M, g^n) system has been shown to have strong stochastic properties. Once the phase variable is, in fact, δ-correlated, the evolution of the coarse-graining function

$$w(u,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\rho(\psi,u,t)d\psi \tag{31}$$

obeys the FPK equation for the u-variable alone. In this case, the distribution function, or rather, the probability density $w(u;t)$, is governed by the FPK equation in the standard form

$$\frac{\partial w(u;t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial u}D_u\frac{\partial w}{\partial u}. \tag{32}$$

Here, $D_u = \langle (u_{n+1} - u_n)^2\rangle / T = Q^2/2T$ is the conventional coefficient of diffusion in phase space

$$D_u = \frac{\omega}{2\Delta\omega}\omega b^2 = \frac{1}{2}\omega^2 P, \tag{33}$$

where $\langle \cdot \rangle$ denotes the phase average and T is the timescale of the problem. The function $w(u,t)$ is a differentiable function that is normalized to 1

$$\int_{u\in\{U\}} w(u,t)du = 1, \tag{34}$$

where $\{U\}$ is a range of the variable u.

For the function $w(u)$ restricted to $[u_a, u_b]$, the invariant distribution can be given by

$$w(u) = (u_b - u_a)^{-1}. \tag{35}$$

The characteristic time for redistribution of u over the spectrum is found to be

$$t_d \simeq 2u_b^2/D_u, \tag{36}$$

which yields $t_d \sim 2 \cdot 10^3$ s.

Let us consider the impact of bouncing on the particle motion. In particular, chaotic motion of particles driven by monochromatic Alfven, or EMIC, waves can arise only as a result of the bounce-effect. This has been shown by Ho *et al.*[23] However, this effect is relatively small and leads to only minor variations of observable quantities near the principal resonance. Consider this effect in detail. Let the condition of wave-particle resonance, $p/m = \omega_B/k_z c$, be valid at a point with the coordinate $z = z_0$ along the line of force. One can examine the particle motion in the vicinity of z_0, taking into account the bounce force, $-\partial H_0/\partial z = -(p_t/2EB)\partial B/\partial z$, $p_t^2 = 2m\omega_B I$, and assuming a parabolic approximation of Earth's magnetic field, $B(z) = B_0(1 + z^2/2l^2)$, where l is the spatial scale for the inhomogeneities. Then the equations of motion are

$$\begin{aligned} \dot{p}_z &= -kb(p_t/E)p_r \sin\psi - (p_t^2/2El^2)(z - z_0), \\ \dot{\psi} &= k_z(p_z - p_r)/E_r, \end{aligned} \tag{37}$$

which are identical to (16), and take into account the bounce-effect. One can then write these equations in an equivalent form

$$\ddot{\psi} + \omega_f^2 \sin\psi + \omega_b^2 \psi = 0, \tag{38}$$

$$\omega_f^2 = k^2(p_t/E)b, \quad \omega_b^2 = (p_t/E)^2/2l^2, \tag{39}$$

where ω_f is the frequency of phase oscillation, ω_b is the bounce frequency, and the initial phase is set to be zero for simplicity. Thus, we learn that the ratio

$$\omega_b^2/\omega_f^2 = (p_t/E)\varepsilon^2/b, \tag{40}$$

where $\varepsilon = 1/kl$, under the resonance EMIC wave-electron interaction, is typically small, no more than 10^{-5}.

According to (40), the bounce effect we speak of is negligible and can be considered as a perturbation of the principal motion. As a result of this, the strongly nonlinear solution of (38) is given by

$$\Delta p/E = \pm 2(\omega_f/kc)\cos(\psi/2) = \pm 2(\omega_f/kc)\cosh^{-1}(\omega_f t) \tag{41}$$

with its peak value $\Delta p/E = \sqrt{(p_t/E)b}\,(\sim 10^{-2})$.

It should be noted that the effect in the quasilinear theory is also treated as a pertubation in calculating bounce-averaging coefficient of diffusion.[24]

Now, it also seems necessary to treat the dynamics of particles in the packet of two EMIC waves with slightly different frequencies. Applying the same method, we obtain the equation

$$\ddot{\psi} + \omega_f^2(1 + \cos\delta\omega t)\sin\psi = 0, \tag{42}$$

where $\delta\omega$ is the difference frequency of the wave packet. This is a well-known equation for a parametric excited nonlinear oscillator that describes the parametric resonance at $\delta\omega \approx \omega_f$.[20] It will cause the overlap of resonances in a region of phase space given by (41).

In a nonlinear wave packet, the overlapping effect arises in a global region of phase space, and the dynamics in this region obey the equation for g^n as above. As the bounce-effect is too small to play an essential role in dynamics, the term, $-\partial H_0/\partial z$, which appears in (8) and describes this effect, averages to zero in the leading order (to first order in ε), as do all other off-resonant terms. In the following order, integrating the term with the help of the relation $dt = (E/p_z)dz$ over the spatial scale of the envelope, we obtain

$$\Delta p_z/p_r = \varepsilon(u^2/2)(L/l). \tag{43}$$

The term is second order in ε and has been dropped in the first of equations of g^n. However, this term can lead to distortions of the phase plane near resonances of the principal frequencies and can be physically associated with the loss of phase coherence.[16] It is easy to see that under these conditions, an additional term proportional to u^2 would appear in the phase advance equation. In our case, the solution of the equations of motion describes a random trajectory, i.e., a trajectory that can be treated as the realization of a random process. It allows us to consider the additional term as an extrinsic noise.

Fluctuations in Earth's magnetic field can also modify the motion of particles. Because the level of fluctuations (background noise) is relatively small, this effect is exhibited in the equations of motion only as a fluctuation of the gyrofrequency ω_B, $\omega_B \to \omega_B(1 + \delta B/B)$. It leads to the appearance the term $\omega_B T\delta B/B$ in phase advance equation. Assuming that these fluctuations are white noise, we include the additional terms in the equations of motion to obtain the closed pair of nonlinear stochastic equations,

$$\begin{aligned} u_{n+1} &= u_n + Q\sin\psi_n, \\ \psi_{n+1} &= \psi_n + \xi_n + (\omega T u_{n+1}^2/2)/(1 - u_{n+1}^2/2) \mod 2\pi, \end{aligned} \tag{44}$$

where the term ξ_n plays the role of a weak stochastic force. The random variable $\xi = \delta b/b$ has a Gaussian probability density

$$p(\xi) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\xi^2/2\sigma^2\right),$$

the mean-square value of which is $\langle \xi_n \xi_{n'} \rangle = \sigma^2 \delta_{nn'}$, where σ^2 is the intensity of noise, and δ is the Kronecker delta.

From Figure 3, we learn that the term ξ_n in the system acts without significantly changing its statistical properties. The effect of extrinsic noise manifests itself in the FPK equation as an additional term in the coefficient of diffusion, $D/D_u = 1\sigma^2/Q^2$. This effect does not significantly affect diffusion induced by stochasticity, due to the global stability of the SA.

IV. PITCH ANGLE SCATTERING AND REP EVENTS

To find what energetic states the particle can be in, we use the invariant of motion to reveal the relationship

$$u^2 = 2\sqrt{e^2 - 1}, \tag{45}$$

where $e = E/E_r$ is the normalized particle energy.

Thus, the measure-preserving transformation $w(e)\, de = w(u)du$, where $w(e)$ is the density of states in energy space, determines this problem completely, subject to appropriate boundary conditions. The corresponding solution for $w(e)$ is found to be

$$w(e) = w(u)e/\sqrt{2}(e^2 - 1)^{3/4}, \quad e \in (e_a, e_b), \quad e = E/E_r. \tag{46}$$

Here, $e_a = \sqrt{1 + u_a^4/4}$ is the minimal value of particle energy, the term $u_a^4/4$ is caused by the wave-particle interaction, and $e_b = 5/4$, $E_b = 2.5\,\text{MeV}$ is the upper boundary of energy spectrum. Finally, by means of (46), we find the mean energy $\sqrt{\langle E^2 \rangle} \approx 2.1\,\text{MeV}$. Then, with the help of (45), we convert equations (19) into the map

$$\begin{aligned} e_{n+1} &= e_n + Q\sqrt{2}(e_n^2 - 1)^{3/4}/e_n \sin\psi_n, \\ \psi_{n+1} &= \psi_n - \omega T\sqrt{e_{n+1}^2 - 1}\Big/\left(1 - \sqrt{e_{n+1}^2 - 1}\right) \bmod 2\pi \end{aligned} \tag{47}$$

to compare result (46) to the values of w_e obtained numerically as a function of e. The prediction (46) is in reasonable agreement with the numerically determined spectrum from the map (47) in the range of interest, as shown in Fig. 4. This behavior of w_e is due to the rate of diffusion in e, $D_e = \text{const.}(e^2–1)^{3/2}/e^2$, which leads to a distribution in the narrow range of energies ~2–2.5 MeV having a sharp maximum at the value close to E_r(2 MeV). Taking account of relation (45), and performing once more the transformation $w(e) = w(u)du/de$ in (31), we derive the following FPK equation for $w(e)$:

$$\frac{\partial w(e,t)}{\partial t} = \frac{\partial}{\partial e} J(e,t), \quad J = \frac{1}{2}\left(D_e \frac{\partial}{\partial e} + \frac{1}{2}\frac{\partial D_e}{\partial e}\right) w, \tag{48}$$

which describes the evolution of $w(e)$. Putting $J = 0$ at the boundary energy spectrum, we obtain the steady-state solution to the equation, which coincides with (46).

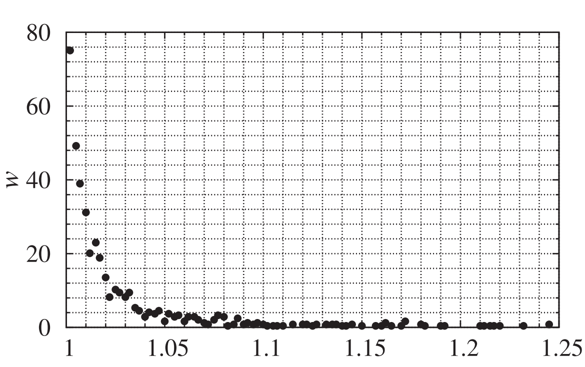

FIG. 4. The energy distribution $w(e)$ with parameter $Q = 0.1$.

Now, we consider how these dynamics are manifested in REP events due to pitch angle scattering. Considering the invariant of motion (20), one defines the pitch angle by

$$\tan\chi = u/(1 - u^2/2). \tag{49}$$

Subject to $\chi_c \ll 1$, where χ_c is the loss cone angle, (49) yields $\chi_c \approx u_c$. So, EMIC waves can scatter into the loss cone angle only those electrons with energies closed to E_r, $E \le E_c$, where

$$E_c = E_r(1 + u_c^4/8). \tag{50}$$

It helps to find the magnitude of the EMIC wave field at which REP events will occur

$$b \ge \sqrt{\Delta\omega/\omega(\omega T)^{3/2}\chi_c^2}, \quad B^w \ge 10^{-3}B(\ge 100\,\text{pT}), \tag{51}$$

where we profit from equations (27) and (20). Carrying through the transformation $w_\chi d\chi = w_u du$ along with (49), where w_χ denotes the function of distribution over χ, we determine the function

$$w_\chi = w_u \frac{\sin^4\chi + \left(1 - \sqrt{1 - \sin^4\chi}\right)^2}{\sin^2\chi + \left(1 - \sqrt{1 - \sin^4\chi}\right)} \cdot \frac{1}{\sin^2\chi} \tag{52}$$

and the coefficient of diffusion in χ,

$$D_\chi = D_u \left(\frac{\sin^2\chi + \left(1 - \sqrt{1 - \sin^4\chi}\right)}{\sin^4\chi + \left(1 - \sqrt{1 - \sin^4\chi}\right)^2}\right)^2 \sin^4\chi. \tag{53}$$

Now, we are capable of interpreting REP events. According to Millan and Thorne,[12] the REP events have a long time burst structure of ~40 min on average with microbursts of ~20 s period, and the energy spectrum of the precipitating electrons is nearly monoenergetic, with its peak at the resonance energy. We build on this work to find out whether the EMIC wave-electron interaction results in electron losses. The functions w_e and w_χ given above are exactly what we want to estimate for the REP events. First, it should be noted that there exists the relationship $w_\chi^2 D_\chi = \text{const}$ which is typical for a dynamical system having a strange attractor.[16] In this case, const $= 2/t_d$, where t_d is defined by 36. From this, it follows that the relaxation time of the pitch angle distribution to a steady state is to be on the order of t_d (10^3 s) the pitch angle diffusion time. Then, noting that the χ-

distribution is in fact almost isotropic, we find the time for diffusion in the loss cone is $t_c = t_d\chi_c^2 (\sim 10\,\mathrm{s})$ at $\chi_c \sim 0.1$. At this rate, the effect is certain to cause a modification of both the pitch angle and energy spectra due to the loss of resonant electrons. Then, the competitive process of diffusion in energy, in turn, must reproduce these distributions, and the recovery time is on the order of t_d, the time of diffusion in pitch angle. In this way, the rate of diffusion in the loss cone directly determines the duration of REP microburst events, and the timescale for temporal structure of REP events has been determined by the recovery time. These results lead to the inference that there is a strong correlation between the processes of energization, angle scattering, and REP events.

Consider the system as an ensemble of particles whose energy and pitch angle spectra are given above. To begin, we define the differential flux of particles

$$dN(\chi, e) = N_r v w_e de \cos\chi w_\chi d\Omega/4\pi, \tag{54}$$

where N_r is the resonant particle density and $d\Omega$ is the element of solid angle.[25] We take into account that the particle's velocity v is very close to the speed of light, and assume that the flux is isotropic along the direction of propagation, $d\Omega = 2\pi\sin\chi d\chi$.

Integrating (54) over the energy spectrum, we find the integral flux of particles with energies $e > e_r$,

$$F(e,\chi) = \frac{1}{2}N_r v \sin^2\chi\left(\frac{e^2-1}{e_b^2-1}\right)^{1/4}. \tag{55}$$

Whence we realize that the flux of precipitating electrons is in fact monoenergetic, which is in agreement with experiments. In keeping with (55), along with (50), we found the fraction of precipitating electrons,

$$J_{loss}/J_{total} = \left(\frac{e_c^2-1}{e_b^2-1}\right)^{1/4}\cdot\chi_c^2 \sim u_c\chi_c^2 \sim \chi_c^3(\sim 10^{-3}). \tag{56}$$

This result can account for observations that the $\sim$1 MeV population is not swept out by storms because of competition between losses and accelerating particles.[26] Based on Lorentzen *et al.*[9] and Millan *et al.*,[10] we estimate the average flux, F_p, of precipitating electrons are $\sim 300\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$. In view of (56), we then write down $F_p = N_r c\chi_c^3$, from which we obtain the estimate $N_r \sim 10^{-5}\,\mathrm{cm}^{-3}$, which appears to be reasonable. Assuming the size of the precipitation region to be approximately $10^{15}\mathrm{cm}^2$, we evaluate a total of 3×10^{22} 2 MeV electrons lost during a one day interval, and the energy input into the atmosphere, to be $6\cdot10^{22}$ MeV/day. In parallel, one can estimate the average rate for energy input in radiation belt (RB) associated with the resonance electron cyclotron heating to be $dE/dt \sim (E-E_r)/t_d$, $dE/dt = 5\times10^{-5}$ MeV/s. Because there are $N_p \sim 3\times10^{22}$ precipitating electrons, $NdE/dt \sim 10^{23}$ MeV/day. The two estimates agree well, and therefore this process is capable of maintaining the population of RB relativistic electrons. This demonstrates that the EMIC wave-electron interaction is among the most likely underlying mechanisms for REP events. Thus, this mechanism may contribute to electron losses; because of its selective nature and low heating rate the contribution to REP events would be relatively small compared to the losses due to chorus-electron interaction. Indeed, the total input from this interaction is two orders smaller than that measured by Imhof and Gaines.[27]

V. CONCLUSION

We have studied the problem of particle scattering and bursty precipitation of relativistic electrons related to stochastic dynamics of RB electrons resonantly interacting with nonlinear EMIC wave. The interaction is shown to result in particle scattering into the loss cone, forming $\sim$10 s microbursts of precipitating electrons. These dynamics can account for the statistical correlations between the processes of energization, pitch angle scattering, and REP events that are manifested on large temporal scales on the order of the diffusion time ($\sim$tens of minutes). Another consequence of this is the nearly monoenergetic spectrum of precipitating electrons. The present study shows that the rate of particle scattering is high enough to support observed fluxes and demonstrates that our approach can be used to interpret REP events.

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